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ON FURSTENBERG'S CHARACTERIZATION OF HARMONIC FUNCTIONS ON SYMMETRIC SPACES

BY

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ABSTRACT

We provide a necessary and sufficient condition on a radial probability measure μ on a symmetric space for which $f = f * \mu$, f bounded, implies that f is harmonic. In particular, we obtain a short and elementary proof of a theorem of Furstenberg which says that if f is a bounded function on a symmetric space which satisfies $f = f * \mu$ for some radial **absolutely continuous** probability measure μ, then f is harmonic.

1. Introduction and preliminaries

Let G be a connected semi-simple Lie group with finite center and K a maximal compact subgroup of G. Let $\mathbf{D} = G/K$ be a symmetric Riemannian space where points of $\mathbf{D}$ are indentified with the cosets gK and functions on G with $f(gk) = f(g)\ \forall k \in K$, are considered as functions on $\mathbf{D}$.

Definition: A function f on $\mathbf{D}$ is harmonic if it satisfies $f(gK) = \int_K f(gkg'K)dk$ for any g, $g' \in G$. That is, the function f admits the mean value property with respect to every K-orbit $\{kg'K : k \in K\}$,

It follows immediately that $\Delta f = 0$ for any G-invariant operator Δ on $\mathbf{D}$ which annihilates constants ([F2], p. 368).

We say that a measure μ on G is radial if $\mu(f(k'gk'')) = \mu(f(g))$ for any continuous function f on G and any k', $k'' \in K$ where $\mu(f(g)) = \int_G f(g)d\mu(g)$. A Lie group G with finite center is simple if its Lie algebra is simple. It is known that every semi-simple Lie group G with finite center is an almost direct product of a finite number of simple Lie groups. That is, $G = G_1 \times G_2 \times \cdots \times G_N$ where G_i are simple, $G_i \cap G_j$ is finite and $g_i g_j = g_j g_i$ for any $g_i \in G_i$, $g_j \in G_j$, $i \neq j$.

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For $g \in G$, let $(g)_i$ denote the i'th component of g.

A by now classical result of Furstenberg says that if f is a bounded function on $\mathbf{D}$ which satisfies the mean value property with respect to a given radial measure μ, then f is harmonic ([F1] and [F2] Theorem 5 (c) $\rightarrow$ (d)). More precisely,

FURSTENBERG'S THEOREM: *If μ is a radial absolutely continuous probability measure on G and f is a bounded function on $\mathbf{D}$ which satisfies*

$$f(g) = \int_G f(gg')d\mu(g') \quad \forall g \in G, \tag{1}$$

then f is harmonic on $\mathbf{D}$.

The result is a main step in the study of the Poisson formula for harmonic functions on symmetric spaces.

For the proof, Furstenberg uses probabilistic methods and his main tool is the theory of martingales. He establishes a more general result (Theorem 3.1 in [F1]) which enables him to study the solutions of (1) (The so-called μ-harmonic functions) for measures μ which are not necessarily radial.

Our main purpose is to provide an elementary "Abelian" proof of Furstenberg's theorem. The essential point is the commutativity of the algebra of radial measures under convolution on G. Our proof enables us to give a necessary and sufficient condition for a radial probability measure μ to characterize harmonic functions on $\mathbf{D}$.

Our main result is the following:

THEOREM: *Let $G = G_1 \times G_2 \times \cdots \times G_N$ be a decomposition of a semi-simple Lie group G of finite center into simple components. Let μ be a radial probability measure on G. Then every bounded function f on $\mathbf{D} = G/K$ which satisfies* (1) *is harmonic if, and only if, the following holds: the semi-group generated by the double-cosets $Kg'K$ in* Supp(μ) *is equal to G.*

(2) *Equivalently, for every j, for which G_j is non-compact, there exists a double-coset $Kg'K$ in* Supp(μ) *such that $(g')_j \notin K_j$, where K_j is a maximal compact subgroup of G_j.*

In particular, if μ is **absolutely continuous** *then μ satisfies* (2), *and Furstenberg's Theorem follows.*

We shall need some more notation. Let $d(g'K, g''K)$ denote the distance between the points $g'K$, $g''K \in \mathbf{D}$ with respect to a fixed group invariant Riemannian metric on $\mathbf{D}$. A function f on $\mathbf{D}$ is radial if

$$f(kgK) = f(gK) \quad \text{for any } k \in K.$$

Let $\tilde{\mu}_{g'}$ denote the K-invariant normalized measure on the K-orbit $\{kg'K: k \in K\}$ in $\mathbf{D}$. Hence,

$$\tilde{\mu}_{g'}(f) = \int_K f(kg'K)dk$$

and

$$(f * \mu_{g'})(gK) = \int_K f(gkg'K)dk \quad \text{where } \mu_{g'}(g) = \tilde{\mu}_{g'}(g^{-1}).$$

If f is radial, then convolution by $\mu_{g'}$ is a "radial translate" of f in the sense that $f * \mu_{g'}$ is also radial and $(f * \mu_{g'})(K) = f(g'K)$. Furthermore, for radial functions f we have

$$(3) \qquad (f * \mu_{g'_1}) * \mu_{g'_2} = (f * \mu_{g'_2}) * \mu_{g'_1}.$$

For $G = \mathrm{SL}(n,\mathbb{R})$ it follows easily, by induction on n, that the semi-group generated by any single double-coset $Kg'K$ where $g' \notin K$, $K = \mathrm{SO}(n)$, is equal to G. That is, the union of the sets K, $Kg'K$, $Kg'Kg'K, \dots$ is equal to G/K. It can be shown that this property is shared by any simple Lie group. Consequently, for $G = G_1 \times G_2 \times \cdots \times G_N$, G_i simple, the semi-group generated by any single double-coset $Kg'K$, where $(g')_j \notin K_j$ for all j for which G_j is non-compact, is equal to G.

Remark: It is convenient to consider K-orbits as Riemannian spheres $\{gK: d(gK, K) = R\}$, which are exactly the same for groups of rank one. However, in the higher rank case, K-orbits are far from being spheres. Actually, in this case, each sphere with $R > 0$ is an uncountable union of disjoint K-orbits.

2. The proof of the theorem

We first claim that if equation (1) has a non-harmonic solution, then it also has a radial non-constant solution. Indeed, if f is non-harmonic then

$$f(g_0K) \neq \int_K f(g_0kg'K)dk \quad \text{for some } g_0, g' \in G.$$

Let $H(gK) = \int_K f(g_0kgK)dk$. Then H is a radial function which satisfies (1) since the space of solutions of (1) is left translation invariant, and $H(K) \neq H(g'K)$.

We can thus assume that f is a radial real-valued, bounded solution of (1) and we need to show that f is constant. By convolving both sides of (1) by a radial integrable function we may further assume that f is uniformly continuous on $\mathbf{D}$.

Let $f_{g'} = f * \mu_{g'} - f$ for some fixed $g' \in G$. Suppose that $\mathrm{Sup}_D f_{g'} = a > 0$ and let $g_n K \in \mathbf{D}$ be such that

$$\lim_{n\to\infty} f_{g'}(g_n K) = a.$$

Let $f_{n,g'} = f_{g'} * \mu_{g_n}$. The sequence $f_{n,g'}$ is uniformly bounded and equicontinuous. Consequently, by Ascoli's theorem there exists a subsequence $f_{n_k,g'}$ that converges uniformly on compact sets to f_0 where $f_0(K) = a$ and $f_0(gK) \leq a$, $\forall g \in G$. Since f_0 satisfies (1) we deduce that $f_0 = a$ on all sets of the form $Kg_1''K$, $Kg_1''Kg_2''K, \ldots, Kg_1''K \cdots g_n''K \cdots$ for $Kg_i''K$ in the support of μ, which by (2) is equal to $\mathbf{D}$. Hence $f \equiv a$ on $\mathbf{D}$. Let ℓ be a positive integer and let $B_{\ell R}$ denote the ball $\{gK\colon d(K, gK) \leq \ell R\}$ where $d(K, g'K) = R$.

Let N be a positive integer for which $f_{N,g'} > a/2$ on $B_{\ell R}$, and let $T = f * \mu_{g_N}$. By (3), we have $f_{N,g'} = (f * \mu_{g_N}) * \mu_{g'} - f * \mu_{g_N} = T * \mu_{g'} - T$, and we know that $f_{N,g'}(K) = (T * \mu_{g'})(K) - T(K) > a/2$.

Since T is radial, $(T * \mu_{g'})(K) = T(g'K)$. But $g'K \in B_{\ell R}$, implying that $f_{N,g'}(g'K) = (T * \mu_{g'})(g'K) - T(g'K) = \int_K T(g'kg'K)dk - T(g'K) > a/2$.

Hence, for some $k_1 \in K$

$$T(g'k_1g'K) - T(g'K) > \frac{a}{2}.$$

Let $g_2' = g'k_1g'$. Since $g_2'K \in B_{\ell R}$, we have $T(g_2'K) > T(g'K) + a/2 > T(K) + a$.

By iteration, we can find $g'K, g_2'K, \ldots, g_\ell'K$ in $B_{\ell R}$ such that

$$T(g_\ell'K) > T(g_{\ell-1}'K) + \frac{a}{2} > T(g_{\ell-2}'K) + a > \cdots > T(K) + \frac{\ell}{2}a.$$

Since ℓ is arbitrary, it follows that $f_{g'} \equiv 0$ for all $g' \in G$, implying that f is constant, which proves the "if" part of the theorem.

To prove the necessity of the condition, suppose that for every double-coset $Kg'K$ in $\mathrm{Supp}(\mu)$ we have $(g')_1 \in K_1 \neq G_1$. Then every function $f(g) = f(g_1, g_2, \ldots, g_N)$ on $\mathbf{D}$, which is constant as a function of $g_2 \cdots g_N$ for every fixed g_1, satisfies (1) but is not necessarily harmonic. This proves the "only if" part and the proof of the theorem is completed.

Remark: Our proof generalizes Furstenberg's result for locally compact separable unimodular groups G and radial probability measures μ which satisfy:

(i) (G, K), where K is a compact subgroup of G, is a Gelfand pair. That is, the algebra of K-bi-invariant integrable functions on G is commutative.

(ii) The semi-group generated by $\mathrm{Supp}(\mu)$ is dense in G.

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